University of California, Berkeley
Physics H7C Fall 2002 (*Strovink*)

SOLUTION TO FINAL EXAMINATION

Problem 1.
(**a**)
The momentum p of the recoiling nucleus balances the momentum E_γ/c of the photon. By conservation of energy,

$$
\begin{aligned}
(M+\Delta M)c^2 &= E_{\text{recoil}} + E_\gamma \\
&= \sqrt{p^2c^2 + M^2c^4} + E_\gamma \\
&= \sqrt{E_\gamma^2 + M^2c^4} + E_\gamma \\
(M+\Delta M)c^2 - E_\gamma &= \sqrt{E_\gamma^2 + M^2c^4} \\
(M^2 + 2\Delta M + \Delta M^2)c^2 - 2(M+\Delta M)c^2E_\gamma + E_\gamma^2 &= E_\gamma^2 + M^2c^4 \\
(2\Delta M + \Delta M^2)c^2 - 2(M+\Delta M)c^2E_\gamma &= 0 \\
(2\Delta M + \Delta M^2)c^2 &= 2(M+\Delta M)c^2E_\gamma \\
\Delta Mc^2\frac{1+\frac{\Delta M}{2M}}{1+\frac{\Delta M}{M}} &= E_\gamma \\
\Delta Mc^2\big(1-\frac{\Delta M}{2M}\big) &\approx E_\gamma\ .
\end{aligned}
$$

The same result may be obtained even more easily by the method of successive approximations: first approximate E_γ by ΔMc^2 and set p to E_γ/c; then approximate the kinetic energy K of the (nonrelativistic) recoil nucleus as $\frac{1}{2}p^2/M = \frac{1}{2}\frac{\Delta M}{M}\Delta Mc^2$; and finally, to conserve energy, correct E_γ downward by K.
(**b**)
If the $(A,Z)'$ energy uncertainty ΔE were zero, the γ would be too weak by an energy of order $\frac{\Delta M}{M}\Delta Mc^2$ to initiate the reverse reaction. For reabsorption to be able to proceed, ΔE must be at least of that same order. Therefore, by the Uncertainty Principle,

$$
\begin{aligned}
\tau &\approx \frac{\hbar}{\Delta E} \\
&\lesssim \frac{\hbar M}{(\Delta M)^2c^2}\ .
\end{aligned}
$$

These equations underly the Mossbauer effect, which was used by Pound and Rebka to first measure the gravitational redshift.

Problem 2.
(**a**)

$$
\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}\ .
$$

The resulting light is plane polarized in the direction $\hat{x}\cos\alpha - \hat{y}\sin\alpha$, *i.e.* the direction of plane polarization is rotated clockwise by the angle α.

(**b**)

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

The resulting light is plane polarized in the $\hat{x}$ direction, *i.e.* the direction of plane polarization again is rotated clockwise by the angle α.

(**c**)

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos\alpha - i\sin\alpha \\ -i\cos\alpha - \sin\alpha \end{pmatrix} \propto \begin{pmatrix} 1 \\ -i \end{pmatrix} .$$

The resulting polarization is unchanged.

(**d**)

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\alpha + i\sin\alpha \\ i\cos\alpha - \sin\alpha \end{pmatrix} \propto \begin{pmatrix} 1 \\ i \end{pmatrix} .$$

The resulting polarization is unchanged.

(**e**)

Light emerging from the vertical ($\hat{y}$) polarizer will be reduced in irradiance by a factor of **two** compared to the unpolarized beam. (This is because E_y doesn't interfere with E_x, so $I = I_x + I_y$; in unpolarized light the two contributions to I are equal.) Thereafter the direction of plane polarization is rotated by $\pi/2$ by the nematic cell, so the beam passes unattenuated through the horizontal ($\hat{x}$) polarizer. After reflection it is still horizontally polarized, so again it passes unattenuated through the $\hat{x}$ polarizer. Finally its polarization is rotated through $-\pi/2$ by the nematic cell, so that it also passes unattenuated through the $\hat{y}$ polarizer. Therefore the only attenuation occurs in the initial traversal of the first polarizer:

$$\frac{I_{\text{emergent}}}{I_{\text{incident}}} = \frac{1}{2} .$$

(To darken an LCD pixel, a voltage is applied to disrupt the operation of the nematic crystal.)

Problem 3.

In analogy to the optical coating, we consider a stepped potential:

$$\begin{aligned} V = 0, \quad x < 0 :& \;\; \hbar^2 k^2 = 2mE \\ V = V_1, \quad 0 < x < x_1 :& \;\; \hbar^2 k_1^2 = 2m(E - V_1) \\ V = V_0, \quad x_1 < x :& \;\; \hbar^2 k_0^2 = 2m(E - V_0) , \end{aligned}$$

where x_1 is the width of the step, V_1 is its height, and the k's are parameters of the time-independent Schroedinger equation solutions $\propto \exp(ikx)$ in the three regions. In the optical problem, the phase velocity of the wave is inversely proportional to n, the refractive index. In the Schroedinger problem, the phase velocity is $\omega/k = E/(\hbar k)$, also inversely proportional to k. So, in analogy to the optical relation $n_1 = \sqrt{n_0 n_2}$, here we have

$$\begin{aligned} k_1^2 &= k k_0 \\ (E - V_1) &= \sqrt{E(E - V_0)} \\ V_1 &= E - \sqrt{E(E - V_0)} . \end{aligned}$$

In the step region $0 < x < x_1$, the Schroedinger wavelength is $\lambda_1 = 2\pi/k_1$. The step width x_1 should be one-quarter of this:

$$\begin{aligned} x_1 &= \frac{\lambda_1}{4} \\ &= \frac{2\pi}{4k_1} \\ &= \frac{2\pi\hbar}{4\sqrt{2m(E-V_1)}} \\ &= \frac{\pi\hbar}{\sqrt{8m\big(E-(E-\sqrt{E(E-V_0)})\big)}} \\ &= \frac{\pi\hbar}{\sqrt{8m\sqrt{E(E-V_0)}}} \,. \end{aligned}$$

Problem 4.
(a)
Using $L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$, and neglecting the term in E relative to the term in L^2, the T.I.S.E. becomes

$$\begin{aligned} 0 &= -\frac{\hbar^2}{2M} b(b-1) r^{b-2} + \frac{\hbar^2 l(l+1)}{2M} r^{b-2} \\ &= -b(b-1) + l(l+1) \\ &= b^2 - b - l(l+1) \\ b &= \frac{1 \pm \sqrt{1+4l(l+1)}}{2} \\ &= \frac{1 \pm (2l+1)}{2} \\ &= -l \text{ or } l+1 \,. \end{aligned}$$

Only the second solution is physical, as only it vanishes at $r = 0$ as must occur when the angular momentum is finite; also, the first solution is unnormalizably infinite at $r = 0$, even for the lowest possible $l = 1$, due to the extra factor of r^{-1} in u. Hence

$$b = l + 1 \,.$$

(b)
Considering the properties of the Y_{lm}'s, only the $l = 0$ eigenfunction is spherically (and therefore cylindrically) symmetric. The $(l > 0, m = 0)$ eigenfunctions are cylindrically symmetric, but for $m \neq 0$ the phase factor $\exp(im\phi)$ violates cylindrical symmetry. Only the $l = 0$ probability density $u^* u$ is both spherically and cylindrically symmetric, while all the other probability densities are only cylindrically symmetric.

Problem 5.

(**a**)

There are four different spin-projection states ($m_s = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$) at each harmonic oscillator energy eigenvalue $E_n = \hbar\omega_0(n + \frac{1}{2})$; the Pauli principle allows each of these states to be occupied by only one fermion. Therefore, as the energy level increases from n to $n+1$, $\Delta N = 4$ new states become available. The change in energy required to go from level n to level $n+1$ is

$$\Delta E = \hbar\omega_0(n + 1 + \tfrac{1}{2}) - \hbar\omega_0(n + \tfrac{1}{2}) = \hbar\omega_0 \ .$$

Therefore

$$\frac{\Delta N}{\Delta E} = \frac{4}{\hbar\omega_0} \ .$$

(**b**)

Because of the factor of 4, $2N_0$ fermions may be accommodated in the lowest $N_0/2$ energy states. Therefore

$$\begin{aligned} n_{\max} &= N_0/2 \\ E_{\max} \equiv E_F &= \hbar\omega_0(n_{\max} + \tfrac{1}{2}) \\ &= \tfrac{1}{2}\hbar\omega_0(N_0 + 1) \ . \end{aligned}$$

(**c**)

In general, at $t = 0$ the particle's wavefunction is a sum over energy eigenstates:

$$\psi(x, t = 0) \equiv u(x) = \sum_{n=0}^{\infty} A_n u_n(x) \ ,$$

where A_n is a complex constant. At a later time t, each term in the sum will acquire a different phase factor

$$\exp(i\phi_n) \equiv \exp(-iE_n t/\hbar) \ .$$

The expectation value $\langle x \rangle$ of the particle's position is

$$\langle x \rangle = \sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} A_m^* u_m^* e^{-i\phi_m} x A_n u_n e^{i\phi_n} \, dx \ .$$

At $t > 0$, the terms in the integral differ from the terms at $t = 0$ by the phase factors $\exp i(\phi_n - \phi_m)$. Only when $\phi_n - \phi_m$ is always equal to a multiple of 2π will every term remain the same. The longest time interval is required for the case $|n - m| = 1$, so that

$$\begin{aligned} |\phi_n - \phi_m| = \frac{(E_n - E_{n-1})t}{\hbar} &= \omega_0 t = 2\pi \\ t &= \frac{2\pi}{\omega_0} \quad \text{(the classical result)} \ . \end{aligned}$$

Problem 6.
Balancing the satellite's mass × centripetal acceleration with the gravitational force on it from the star, we solve for its (nonrelativistic) velocity^2 in circular orbit:

$$\begin{aligned} \frac{mv^2}{R} &= \frac{GMm}{R^2} \\ v^2 &= \frac{GM}{R} \ . \end{aligned}$$

Since the light pulse moves purely in the radial direction and the satellite moves purely in the tangential direction, the satellite is neither approaching nor receding from the observer when the pulse is emitted (and the problem says nothing about possible Hubble expansion of the star-observer distance). Therefore the only part of the relativistic Doppler shift that applies is the usual time dilation: the observed frequency is

$$\begin{aligned} \omega_{\rm obs} &= \frac{\omega_0}{\gamma} \\ &= \omega_0 \sqrt{1 - \frac{v^2}{c^2}} \\ &= \omega_0 \sqrt{1 - \frac{GM}{c^2 R}} \\ &\approx \omega_0 \big(1 - \frac{GM}{2c^2 R}\big) \\ \frac{\Delta\omega_D}{\omega} &\approx -\frac{GM}{2c^2 R} \ . \end{aligned}$$

As for the gravitational redshift, relative to $R = \infty$, the photon gains extra energy $GMm_{\rm eff}/R$ in the attractive potential of the star. Here $m_{\rm eff} = E/c^2$ and $E = \hbar\omega$ is the photon's energy. It loses this extra energy after climbing from radius R to the effectively infinite radius of the observer. Therefore

$$\begin{aligned} \Delta E = \hbar \Delta\omega_G &= -\frac{GM\hbar\omega}{c^2 R} \\ \frac{\Delta\omega_G}{\omega} &= -\frac{GM}{c^2 R} \ . \end{aligned}$$

Finally, ordinary differential calculus relates $\Delta\omega$ to $\Delta\lambda$:

$$\begin{aligned} \lambda &= \frac{2\pi c}{\omega} \\ d\lambda &= -\frac{2\pi c}{\omega^2} d\omega \\ &= -\lambda \frac{d\omega}{\omega} \\ \frac{d\lambda}{\lambda} &= -\frac{d\omega}{\omega} \\ \frac{\Delta\lambda}{\lambda} &\approx -\frac{\Delta\omega}{\omega} \ . \end{aligned}$$

Therefore

$$\frac{\Delta\lambda_G}{\Delta\lambda_D} = \frac{\Delta\omega_G}{\Delta\omega_D} \approx \frac{GM/c^2 R}{GM/2c^2 R} = 2 \ .$$